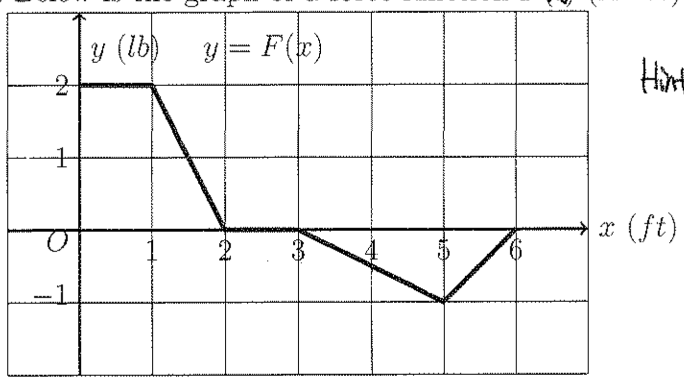
1st (Hint):

Practice Midterm1 (Take-home quiz 6, due on Tue, Oct 11)

Q1 Below is the graph of a force function $F(x)$ (in lbs).



Hint: $W = \int_a^b F(x)\cdot dx$.

work done by $F(x)$ moving an object from $x=a$ to $x=b$

The definite integral represents the AREA of the region below the curve of $F(x)$.

(a) How much work is done by the force in moving an object from $x = 0$ to $x = 3$?

Caution: The AREA counts sign. (It ~~can~~ is ~~be~~ negative for region below x-axis.

(b) How much work is done by the force in moving an object from $x = 0$ to $x = 5$?

Q2 Find the derivatives of the following functions

(a) $f(x) = [\tan^{-1} x]^{\ln(2x)}$, find $f'(x)$

Hint: log-differential rule $y = (\tan^{-1}x)^{\ln(2x)} \Rightarrow \ln y = \ln[\tan^{-1}x]^{\ln(2x)} = \ln(2x)\cdot\ln[\tan^{-1}x]$

Then take derivative both sides.

(b) $f(x) = \pi^{\sin^{-1}(\sqrt{x})}$, find $f'(x)$

Hint: $\frac{d}{dx} a^x = \ln a \cdot a^x$, where $a = \pi$ in this problem.

And then apply chain-rule.

(c) $f(x) = x^2 + 3\sin x + 1$, find $(f^{-1})'(1)$

Hint: $(f^{-1})'(a) = \dfrac{1}{f'[f^{-1}(a)]}$ where $f(0) = 1$ and $f^{-1}(1) = 0$.

Q3 Determine whether the following limits exist or not. Find the limit if it exists.

(a)

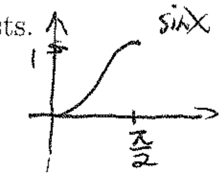
$$\lim_{\theta\to\pi/2}\frac{1-\sin\theta}{1+\cos(2\theta)}$$

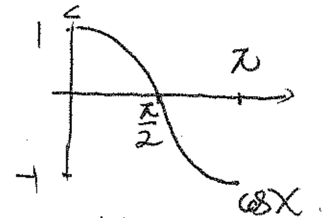
Hint: $1-\sin\frac{\pi}{2}=1-1=0$

$1+\cos(2\cdot\frac{\pi}{2})=1-1=0$

$\sin\frac{\pi}{2}=1$

$\cos\pi=-1$





$\frac{0}{0}$ type: Apply L'Hospital Rule.

(b)

$$\lim_{x\to+\infty}\sqrt{x}\cdot e^{-x/2}$$

Hint: $e^{-\infty}=0$, $\infty\cdot 0$ type, use 'flip-Trick'

$\infty\cdot 0 \xrightarrow{\text{flip}} \frac{\infty}{\frac{1}{0}}$

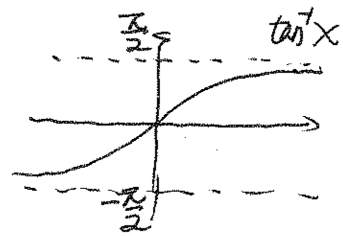
Hint: $e^{-A}=\frac{1}{e^{A}}$

(c)

$$\lim_{t\to 0^+} t\cdot\tan^{-1}(1/t)$$

Hint: $\tan^{-1}(\frac{1}{0^+})=\tan^{-1}(+\infty)=\frac{\pi}{2}$

Caution: NOT Indeterminate Type



Q4 Evaluate the following integrals

(a)

$$\int \sin(3t)\cdot(2t+1)\mathrm{d}t$$

Hint: Poly × sin $\rightarrow$ IBP typical type

$u=$ Poly

$dv=\sin\cdot dt$

(b)

$$\int_0^\infty 2^{-x}\mathrm{d}x$$

Hint: $2^{-x}=(2^{-1})^{x}$. Apply the formula $\int a^x\,dx$ with $a=2^{-1}=\frac{1}{2}$.

(c)

$$\int\frac{5}{\sqrt{9-25x^2}}\mathrm{d}x$$

Hint: DO NOT use Trig-Sub

Use general u-sub and the given formula of

$$\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x.$$

Q4 Evaluate the following integrals.

(a)

$$\int \frac{x^2}{(x^2+1)^{5/2}}\mathrm{d}x$$

Hint: $x^2+1 \longleftrightarrow \tan^2\theta + 1 = \sec^2\theta$

Trig-Sub: $x = \tan\theta$.

Hint2: Use Trig-Relation $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\sec\theta = \frac{1}{\cos\theta}$ to simplify the trig-integral.

Hint3: "ODD rule" for sin-cos product.

(b)

$$\int_8^\infty \frac{10}{x^2-4x-21}\mathrm{d}x$$

Hint: $x^2-4x-21 = (x-7)\cdot(x+3)$.

P.F.D. formula for two distinct linear factors.

$$\frac{*}{(x-a)\cdot(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

Hint2: $\int \frac{1}{x-a}dx = \ln|x-a| + C$

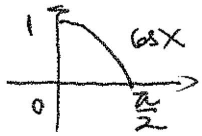
Hint3: $\ln\square - \ln\triangle = \ln\frac{\square}{\triangle}$

Hint4: $\lim_{t\to\infty} \ln ☆ = \ln \lim_{t\to\infty} ☆$

Q5 The solid is generated by revolving the curve $y = \cos x - 1$ for $0 \le x \le \pi/2$ about the axis $y = 1$.

(a) Sketch the solid and set up an integral for the volume of it.

Hint: graph of $\cos x$

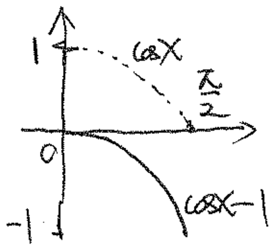


Hint2: $V = \int_a^b A(x) \cdot dx$.

$A(x)$: area of cross-section

graph of $\cos x - 1$

shift 1-unit down



Rotating solid: $A(x) = \pi \cdot R^2$

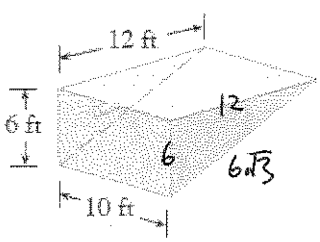
R: distance between the curve and the axis.

(b) Find the volume of the rotating solid.

Hint: double angle formula $\cos^2 \square = \frac{1+\cos 2\square}{2}$

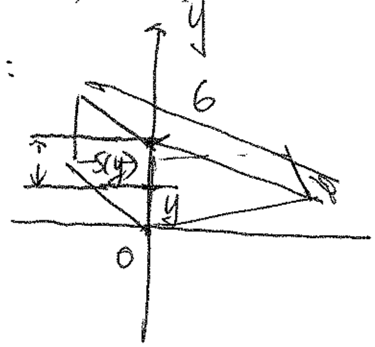
linear formula for cos: $\int \cos(ax)\,dx = \frac{1}{a} \cdot \sin(ax) + C$.

Q6 A tank (shown below) is full of oil weighing 10 lb/ft^3. Find the work required to pump the oil out of the spout. The base is a $10 \times 6\sqrt{3}$ rectangle. The back end is a 6×10 rectangle, the two sides are right triangle with height 6, base $6\sqrt{3}$ and hypotenuse 12 (all in ft). The spout is 6 ft from the base.



Hint2:

s(y):



Hint:

Water-pumping formula:

$W = \int_0^a 6 \cdot s(y) \cdot A(y)\,dy$

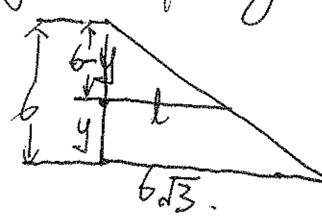
6: weighing-density

s(y): distance to target height at y.

A(y): Area of cross-section.

Hint3: Cross-Section at y is a rectangle.

$6 \times l$



Similar triangle: $\frac{l}{6\sqrt{3}} = \frac{6-y}{6}$

Build your OWN!

Integrals

- **Volume**: Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x-axis, then the volume of S is given by

Rotating case:

$A(x) = \pi \cdot [f(x) - \text{AXIS}]^2$ $$V = \int_a^b A(x)\, dx$$

- **Work**: Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:

Water-Pumping:

$W = \int_a^b 6 \cdot s(y) \cdot A(y)\, dy$ $$W = \int_a^b f(x)\, dx$$

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

$(\frac{1}{x})' = -\frac{1}{x^2}$

- $\int \frac{1}{x}\, dx = \ln|x| + C$, $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$ $n \neq -1$
- $\int \tan x\, dx = \ln|\sec x| + C$, $(\tan x)' = \sec^2 x$, $(\cot x)' = -\csc^2 x$

$(\sec x)' = \tan x \cdot \sec x$, $(\csc x)' = -\cot x \cdot \csc x$.

- $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

$(a^x)' = \ln a \cdot a^x$

- $\int a^x\, dx = \frac{a^x}{\ln a} + C$ for $a \neq 1$

$a^x = e^{\ln a \cdot x}$

$a^{-x} = \frac{1}{a^x}$, $a^x = \frac{1}{a^{-x}}$.

- **Integration by Parts**:

$$\int u\, dv = uv - \int v\, du$$

$\int \underbrace{\text{Poly}}_{u} \times \underbrace{\exp/\sin x/\cos x\ dx}_{dv}$

$dv = e^{ax} dx \Rightarrow v = \int e^{ax} dx = \frac{1}{a} e^{ax}$

$dv = \sin(ax) dx \Rightarrow v = \int \sin(ax) dx = -\frac{1}{a}\cos(ax)$

$dv = \cos(ax) dx \Rightarrow v = \int \cos(ax) dx = \frac{1}{a}\sin(ax)$

- limits: $e^{+\infty} = +\infty$, $e^{-\infty} = 0$, $\ln\infty = \infty$, $\ln 0^+ = -\infty$

$\tan^{-1}(\pm\infty) = \pm\frac{\pi}{2}$. $e^0 = 1$, $\ln 1 = 0$

- L'Hopital Rule: $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

$\frac{\infty}{\infty}$, $\frac{0}{0}$, $0 \cdot \infty = \frac{0}{\frac{1}{\infty}} = \frac{\infty}{\frac{1}{0}}$

- P.F.D. $\frac{*}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$, $\frac{*}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

$\int \frac{1}{x-a} dx = \ln|x-a|$, $\int \frac{1}{(x-a)^2} dx = -\frac{1}{x-a}$.

Derivatives

- $\frac{d}{dx}(\sinh x) = \cosh x$ $\quad \frac{d}{dx}(\cosh x) = \sinh x$
- Inverse Trigonometric Functions: $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$, $\int \frac{dx}{\sqrt{a^2 - b^2x^2}}$,

u-Sub: $bx = au$

$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}}$ $\quad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$\boxed{\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}}$ $\quad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}(\frac{x}{a})$

- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Hyperbolic and Trig Identities

- Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh x}$$

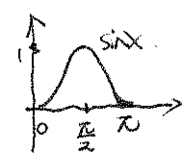
$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh x}$$

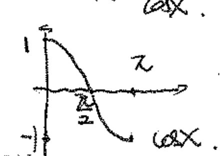
$$\tanh(x) = \frac{\sinh x}{\cosh x} \qquad \coth(x) = \frac{\cosh x}{\sinh x}$$

- $\cosh^2 x - \sinh^2 x = 1$
- $\boxed{\cos^2 x + \sin^2 x = 1}$ $\tan^2 x + 1 = \sec^2 x$, $\tan x = \frac{\sin x}{\cos x}$,

$\sec x = \frac{1}{\cos x}$.

- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2\sin x \cos x$





- $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$
- Trig-Sub: $\sqrt{a^2 - b^2x^2}$ $\quad bx = a\sin\theta$

$\sqrt{a^2 + b^2x^2}$ $\quad bx = a\tan\theta$

$\sqrt{b^2x^2 - a^2}$ $\quad bx = a\sec\theta$.